SOLVENSWISS TECHNICAL DOCUMENT

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2 INTRODUCTION

This document discusses the mathematics and formulas used in the Solvenswiss methodology. In the following, a pension fund is analyzed in view of its liabilities. The aim of the Solvenswiss is to extend the technical analysis, usually performed within the pension fund's annual report, to a risk analysis. This latter analysis aims to incorporate, as much as possible, the market risk and the cash flows risk, which are due to uncertainty about market and liability conditions in the future.

Section 3 introduces briefly the technical analysis. Section 4 discusses at length the risk analysis and is the main contribution of this document. Section 5 presents a computation of the fluctuation reserves objective.

3 TECHNICAL ANALYSIS

3.1 LIABILITIES AND TECHNICAL LIABILITIES

The liabilities consist in the pension fund's commitment to its members of paying them every year t a given cash flow C_t. The cash flow C_t is the members' claim at year t on the assets of the pension fund. As the active members are contributing to the pension fund while the passive members are receiving their pensions, the cash flow may be negative, if the contributions exceed the pensions, or positive, if the pensions exceed the contributions. A particular year is a date in the future and we consider $t,\ldots,T$, the T future years. In general, for a closed pension fund, T is of the order of 80 to 100 years. As the cash flow C_t is a forecast of a future cost, it is uncertain. In what follows C_t is the expected value of the cash flow. The technical liabilities are the sum of the discounted cash flows defined as:

$$L = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}. \tag{3.1}$$

The discount rate r is referred to as the technical rate. The interpretation of the technical rate is the return that the pension fund offers to its members on their capital. Indeed, if C_t is promised today in t years, then the value the pension fund should put aside to cover this commitment is $C_t/(1+r)^t$, assuming that its portfolio has a yearly return of r every year. Therefore, $C_t/(1+r)^t$ is the value that the pension fund should ask to its members today, committing itself to pay it back in t years plus a yearly interest of r.

The fortune F is the market value of the assets of the pension fund. The funding ratio ρ is defined as the ratio between the fortune F of the pension fund and its liabilities L:

$$\rho = F/L. \tag{3.2}$$

The funding ratio is a measure of the health of the pension fund.

3.2 EXAMPLE

We consider a very simple hypothetic pension fund whose liabilities end in five years. Let the technical rate be $r = 3\%$. Then the technical liabilities can be computed according to the following table.

TABLE 3.1

t	r	C_t	$\frac{C_t}{(1+r)^t}$	$\sum_{t=1}^{T}\frac{C_t}{(1+r)^t}$
1	3%	100	97.09	97.09
2	3%	80	75.41	172.50
3	3%	60	54.91	227.40
4	3%	40	35.54	262.94
5	3%	20	17.25	280.20

Therefore the technical liabilities equal 280.20. If the fortune is $F = 285$ then the funding ratio is $\rho = 285/280.2 = 101.7\%$.

The number of funded years N is the latest year at which the fortune exceeds the sum of the discounted cash flows:

$$N = \sup\left\{0 \le t \le T : \sum_{h=1}^{t} \frac{C_h}{(1+r)^h} \le F\right\}. \tag{3.3}$$

When the funding ratio is below 100% then N is the number of years before the fund's bankruptcy. If the funding ratio is above 100%, then N equals T.

3.3 Example

In the previous example, as the funding ratio is larger than 100% then $N = 5$. If $F = 250$ then the funding ratio is 89.2% and the number of funded years equals 3.

3.4 Measure of Sensitivities

The (modified) duration D measures the first order sensitivity of the liabilities to the technical rate. It is defined as the derivative of the logarithm of the liabilities:

$$D = \frac{\partial \ln L}{\partial r} = -\frac{1}{L}\frac{\partial L}{\partial r}. \tag{3.4}$$

The duration is estimated by the effective duration D_{eff} based on three technical rates: $r_- < r_0 < r_+$:

$$D_{\text{eff}} = -\frac{1}{L_0}\frac{L_+ - L_-}{r_+ - r_-}. \tag{3.5}$$

The convexity C is the sensitivity of the duration to the technical rate. It is a measure of second order sensitivity of the liabilities to the technical rate:

$$C = \frac{1}{L}\frac{\partial^2 L}{\partial r^2}. \tag{3.6}$$

The convexity C_{eff} is estimated by the numerical derivative:

$$C_{\text{eff}} = \frac{1}{L_0}\frac{L_+ + L_- - 2L_0}{\left(\frac{r_+ - r_-}{2}\right)^2}. \tag{3.7}$$

The duration and the convexity describe the dependence of the liabilities to the technical rate. A large dependence implies a higher risk on the liabilities. Indeed, in this case, any deviation of the portfolio return from the technical rate implies a miscalculation of the liabilities.

3.5 Example

With a technical rate r_0 at 3%, L_0 is equal to 280.20. Set r_+ at 4% and r_- at 2%, then L_+ is equal to 274.09 and L_- is equal to 286.54. The estimate of the modified duration is therefore

$$D_{\text{eff}} = \frac{-1}{280.20}\frac{274.09 - 286.54}{0.04 - 0.02} = 2.22,$$

while the convexity estimate is

$$C_{\text{eff}} = \frac{1}{280.20}\frac{274.09 + 286.54 - 2 \times 280.20}{\left(\frac{0.04 - 0.02}{2}\right)^2} = 8.53.$$

3.6 The Equilibrium Rate

Using the duration and the convexity, the technical liabilities can be approximated to the second order:

$$L \approx L_0 - D_0\left(r - r_0\right) + \tfrac{1}{2} C_0 L_0 \left(r - r_0\right)^2. \tag{3.8}$$

This approximation enables us to obtain an estimate of the rate r^* such that $L^* = F$, in other words such that the funding ratio equals 100%:

$$r^* = r_0 - \frac{-D_0 + \sqrt{D_0^2 - 2C_0 + 2C_0 F/L_0}}{C_0}. \tag{3.9}$$

This rate is such that the sum of all discounted cash flows equals the fortune. It is therefore an equilibrium rate which may be interpreted as an indicator of health for the pension fund. Indeed, a pension fund with a low equilibrium rate does not need to achieve a high portfolio return years after years to cover its liabilities. Thus, it is an interesting alternative to the funding ratio which depends on the technical rate being itself a rather arbitrary number.

3.7 Example

Following the previous example, if the fortune is equal to 285 then the equilibrium rate is 2.24%:

$$r^* = 0.03 - \frac{-2.22 + \sqrt{2.22^2 - 2\times 8.53 + 2\times 8.53 \times 285/280.20}}{8.53} = 2.24\%.$$

If the fortune is equal to 270, then the equilibrium rate is 4.69%:

$$r^* = 0.03 - \frac{-2.22 + \sqrt{2.22^2 - 2\times 8.53 + 2\times 8.53 \times 270/280.20}}{8.53} = 4.69\%.$$

4 Risk Analysis

4.1 Stochastic Liabilities

The main object of the risk analysis is the liability function

$$L(R,C) = \sum_{t=1}^{T} \frac{C_t}{\prod_{i=1}^{t}\left(1 + R_i\right)}, \tag{4.1}$$

where $R = \left(R_1, \ldots, R_n\right)'$ is the vector of future portfolio returns (at time $t = 1, \ldots, T$) and $C = (C_1, \ldots, C_T)'$ is the vector of the future cash flows. If the future portfolio returns equal R and if the future cash flows equal C, then L is the value that should be invested in the portfolio to cover, years after years, the liabilities of the pension fund. Naturally R and C are not known, thus L is not known neither. In other words, the estimation of the liabilities carries with it two source of risk:

1. the market risk due to uncertainty on R; and
2. the liability risk due to uncertainty on L.

Although unknown, some knowledge about what R and C could be gained from past observations and a probabilistic study.

The difference between the liabilities issued from the technical analysis and the liabilities issued from the risk analysis can be reduced to the risk on returns: if the future portfolio returns are for sure all equal to the technical rate, then these two functions are themselves equal. Obviously, the technical analysis ignores the risk relating to poor forecasting of future returns and future cash flows.

4.2 EXAMPLE

The two tables below exhibit two scenarios for future cash flows and returns. In the first case, the fortune needed today to cover the cash flows over the whole period equals 283.05. In the second case, it equals 293.56.

TABLE 4.1

t	R_t	C_t	$\prod_{i=1}^{t}(1+R_i)$	$\frac{C_t}{\prod_{i=1}^{t}(1+R_i)}$	$\sum_{j=1}^{t}\frac{C_j}{\prod_{i=1}^{j}(1+R_i)}$
1	5.00%	100	1.05	95.24	95.24
2	-2.00%	80	1.03	77.75	172.98
3	10.00%	60	1.13	53.01	225.99
4	-8.00%	40	1.04	38.41	264.40
5	3.00%	20	1.07	18.65	283.05

TABLE 4.2

t	R_t	C_t	$\prod_{i=1}^{t}(1+R_i)$	$\frac{C_t}{\prod_{i=1}^{t}(1+R_i)}$	$\sum_{j=1}^{t}\frac{C_j}{\prod_{i=1}^{j}(1+R_i)}$
1	7.00%	102	1.07	95.33	95.33
2	-7.00%	75	1.00	75.37	170.70
3	4.00%	55	1.03	53.15	223.84
4	-1.00%	43	1.02	41.97	265.81
5	2.00%	29	1.05	27.75	293.56

Therefore, it is not because the future returns and cash flows are unknown that nothing can be said about the liabilities. Indeed, from past observations, one can give information (best estimate from current knowledge) about what L could be by computing the probabilistic distribution of L. In the following, some characteristics of this distribution enabling to compute risk measures on the liabilities are displayed.

4.3 THE EXPECTED VALUE OF L

The second order Taylor expansion is used to describe the sensitivity of the liabilities to the portfolio returns and to the cash flows:

$$\begin{aligned} L(R,C) \approx\ & L(\mu_R,\mu_C) + \nabla L(\mu_R,\mu_C)'(R'-\mu_R', C'-\mu_C')' \\ & +\frac{1}{2}(R'-\mu_R', C'-\mu_C')\nabla^2 L(\mu_R,\mu_C)(R'-\mu_R', C'-\mu_C')' \end{aligned} \tag{4.2}$$

with $\mu_R = E(R)$ and $\mu_C = E(C)$, being the expected values of the returns and the cash flows. Here ∇ and ∇^2 are the derivative operators. Computing the expected value of this approximation[2], one obtains:

$$E\left[L(R,C)\right] \approx L(\mu_R,\mu_C) + \frac{1}{2}E\left[(R'-\mu_R', C'-\mu_C')\nabla^2 L(\mu_R,\mu_C)(R'-\mu_R', C'-\mu_C')'\right] \tag{4.3}$$

Since the rightmost term from the right-hand side of the equation is a scalar, it is equal to its trace. The fact that the trace is linear implies that it permutes with the expectation operator. Furthermore, as the trace is cyclic[3], one obtains:

[2] Using a first order approximation would end up to considering the Jensen inequality as an equality.

[3] $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$

$$\begin{aligned}
&E\left[(R'-\mu_R',C'-\mu_C')\nabla^2 L(\mu_R,\mu_C)(R'-\mu_R',C'-\mu_C')'\right] \\
&=\mathrm{Tr}\left\{E\left[(R'-\mu_R',C'-\mu_C')\nabla^2 L(\mu_R,\mu_C)(R'-\mu_R',C'-\mu_C')'\right]\right\} \\
&=E\left\{\mathrm{Tr}\left[(R'-\mu_R',C'-\mu_C')\nabla^2 L(\mu_R,\mu_C)(R'-\mu_R',C'-\mu_C')'\right]\right\} \\
&=E\left\{\mathrm{Tr}\left[(R'-\mu_R',C'-\mu_C')'(R'-\mu_R',C'-\mu_C')\nabla^2 L(\mu_R,\mu_C)\right]\right\} \\
&=\mathrm{Tr}\left\{E\left[(R'-\mu_R',C'-\mu_C')'(R'-\mu_R',C'-\mu_C')\nabla^2 L(\mu_R,\mu_C)\right]\right\} \\
&=\mathrm{Tr}\left\{\Sigma\nabla^2 L(\mu_R,\mu_C)\right\}
\end{aligned} \tag{4.4}$$

where

$$\Sigma=\begin{pmatrix}\Sigma_{R,R} & \Sigma_{R,C}\\ \Sigma_{C,R} & \Sigma_{C,C}\end{pmatrix},$$

and corresponds to the variance matrix with elements corresponding to $\mathrm{cov}(R_t,R_{t'})$, $\mathrm{cov}(C_t,C_{t'})$ and $\mathrm{cov}(R_t,C_{t'})$. Finally,

$$E\left[L(R,C)\right]\approx L(\mu_R,\mu_C)+\tfrac{1}{2}\mathrm{Tr}\left\{\Sigma\nabla^2 L(\mu_R,\mu_C)\right\}. \tag{4.5}$$

In other words, the expected value of the liabilities is approximated by the liabilities evaluated at the expected returns and the expected cash flows, plus a corrective term implying the covariance matrix of the risk factors (market indices and cash flows). In a next section, this expression will be simplified further.

4.4 Variance Matrix of L

By using the first order Taylor expansion,

$$L(R,C)\approx L(\mu_R,\mu_C)+\nabla L(\mu_R,\mu_C)'(R'-\mu_R',C'-\mu_C')'. \tag{4.6}$$

The variance is therefore approximated[4] by

$$\begin{aligned}
\mathrm{Var}\left\{L(R,C)\right\}&\approx\nabla L(\mu_R,\mu_C)'\mathrm{Var}(R',C')\nabla L(\mu_R,\mu_C)\\
&=\nabla L(\mu_R,\mu_C)'\Sigma\nabla L(\mu_R,\mu_C).
\end{aligned} \tag{4.7}$$

The variance of the liabilities is therefore approximated by a quadratic product implying the covariance matrix of the risk factors.

4.5 Hypotheses about Variance and Correlation Matrices

In what follows, the variance matrix Σ is assumed to be diagonal. This implies that

1. there is no correlation between the portfolio returns and the cash flows;
2. there is no correlation between the portfolio returns at time t and at time t';
3. there is no correlation between the cash flows at time t and at time t'.

Hypotheses 1 and 3 are the weakest. Hypothesis 2 is more debatable. It may be justified by the fact that one year is a long period. An important point is that one should not confound this hypothesis and the non-stationarity hypothesis. In this later case, the expected value and the variance vary in time and, by simulation, already integrate autoregression and heteroskedastic effects.

The fact that Σ is diagonal implies some simplifications in the computations of the expected value and the variance. For the expected value, one can write:

[4] If the second order Taylor expansion was used, then third and forth moments of the distributions of R and C should be used.

$$\mathrm{Tr}\left\{\Sigma\nabla^2 L\right\} = \mathrm{Tr}\left\{\Sigma_R \nabla^2{}_L L\right\} + \mathrm{Tr}\left\{\Sigma_C \nabla_C^2 L\right\} = \sum_{t=1}^{T} \Sigma_R^{(t)} \frac{\partial^2 L}{\partial R_t^2} + \sum_{t=1}^{T} \Sigma_C^{(t)} \frac{\partial^2 L}{\partial C_t^2}, \tag{4.8}$$

where $\sum_R^{(t)} = \sigma^2(R_t)$ and $\sum_R^{(t)} = \sigma^2(C_t)$ are the variances of R_t and C_t, respectively. For the variance, one can write:

$$\begin{aligned} \mathrm{Var}\left\{L(R,C)\right\} &\approx \nabla L(\mu_R,\mu_C)'\Sigma\nabla L(\mu_R,\mu_C) \\ &= \nabla_R L'\Sigma_R \nabla_R L + \nabla_C L'\Sigma_C \nabla_C L \\ &= \sum_{t=1}^{T} \Sigma_R^{(t)} \left(\frac{\partial L}{\partial R_t}\right)^2 + \sum_{t=1}^{T} \Sigma_C^{(t)} \left(\frac{\partial L}{\partial C_t}\right)^2. \end{aligned} \tag{4.9}$$

4.6 Computation of the Derivatives

In order to apply the Taylor expansion, the derivatives of the liabilities with respect to the risk factors have to be computed. The first derivative with respect to the returns is

$$\frac{\partial L}{\partial R_j} = \frac{\partial}{\partial R_j} \sum_{t=j}^{T} \frac{C_t}{\prod_{i=1}^{t}(1+R_i)} = -\frac{1}{(1+R_j)} \sum_{t=j}^{T} \frac{C_t}{\prod_{i=1}^{t}(1+R_i)}. \tag{4.10}$$

The second derivative is

$$\frac{\partial^2 L}{\partial R_j^2} = -\frac{2}{(1+R_j)^2} \sum_{t=j}^{T} \frac{C_t}{\prod_{i=1}^{t}(1+R_i)}. \tag{4.11}$$

The derivative with respect to the cash flows are

$$\frac{\partial L}{\partial C_j} = \frac{\partial}{\partial C_j} \sum_{t=1}^{T} \frac{C_t}{\prod_{i=1}^{t}(1+R_i)} = \frac{C_t}{\prod_{i=1}^{j}(1+R_i)} \tag{4.12}$$

and

$$\frac{\partial^2 L}{\partial C_j^2} = 0. \tag{4.13}$$

4.7 Example

The table contains the hypotheses on moments of cash flows and portfolio returns.

Table 4.3

t	$\mu(C_t)$	$\sigma(C_t)$	$\mu(R_t)$	$\sigma(R_t)$
1	100	5	3.0%	4.0%
2	80	4	5.0%	6.0%
3	60	3	5.5%	7.0%
4	40	2	5.5%	7.0%
5	20	1	5.5%	7.0%

The liabilities evaluated at the expected value is

$$L(\mu_R,\mu_C) = \frac{100}{1.03} + \frac{80}{1.03\times 1.05} \ldots + \frac{20}{1.03\times\ldots\times 1.055} = 272.62.$$

The liabilities are computed in the following table.

TABLE 4.4

t	$\prod_{j=1}^{t}(1+R_j)$	$\frac{C_t}{\prod_{j=1}^{t}(1+R_j)}$	$\frac{\partial L}{\partial R_t}$	$\frac{\partial^2 L}{\partial R_t^2}$	$\frac{\partial L}{\partial C_t}$	$\frac{\partial^2 L}{\partial C_t^2}$
1	1.03	97.09	–264.68	513.95	0.97	0.00
2	1.08	73.97	–167.18	318.43	0.92	0.00
3	1.14	52.59	–96.27	182.50	0.88	0.00
4	1.20	33.23	–46.43	88.01	0.83	0.00
5	1.27	15.75	–14.93	28.30	0.79	0.00

These derivatives are weighted by the variances of each risk factor.

TABLE 4.5

t	$\sigma^2(R_t)\frac{\partial^2 L}{\partial R_t^2}$	$\sigma^2(C_t)\frac{\partial^2 L}{\partial C_t^2}$	$\sigma^2(R_t)\left(\frac{\partial L}{\partial R_t}\right)^2$	$\sigma^2(C_t)\left(\frac{\partial L}{\partial C_t}\right)^2$
1	0.82	0.00	112.09	23.56
2	1.15	0.00	100.61	13.68
3	0.89	0.00	45.41	6.91
4	0.43	0.00	10.56	2.76
5	0.14	0.00	1.09	0.62

Therefore

$$\mathrm{Tr}\left\{\Sigma\nabla^2 L\right\} = \sum_{t=1}^{T}\Sigma_R^{(t)}\frac{\partial^2 L}{\partial R_t^2} + \sum_{t=1}^{T}\Sigma_C^{(t)}\frac{\partial^2 L}{\partial C_t^2} = 0.82 + \ldots + 0.14 + 0.00 + \ldots + 0.00 = 3.43.$$

The expected value of the liabilities is therefore

$$E(L) \approx 272.62 + \frac{3.43}{2} = 274.34.$$

Furthermore

$$\mathrm{Var}(L) = 112.09 + \ldots + 1.09 + 23.56 + \ldots + 0.62 = 317.31$$

and

$$\sigma(L) = \sqrt{317.31} = 17.81.$$

4.8 CONTRIBUTIONS TO VARIANCE

The aim is to decompose the variance into contributions coming from

1. each asset class;
2. each cash flows;
3. each time period.

It was previously established (3.9) that

$$\mathrm{Var}\left\{L(R,C)\right\} \approx \sum_{t=1}^{T}\Sigma_R^{(t)}\left(\frac{\partial L}{\partial R_t}\right)^2 + \sum_{t=1}^{T}\Sigma_C^{(t)}\left(\frac{\partial L}{\partial C_t}\right)^2.$$

From this decomposition of the variance into a sum, one can already obtain the contribution to the variance of the portfolio return at year t and of the cash flow at year t. However, one can be more precise than this, and obtain the contribution of each asset class. First the portfolio return can be written as

$$R_t = \alpha' R_t^{(\cdot)}, \tag{4.14}$$

where $\alpha = (\alpha_1, \ldots, \alpha_p)'$ is the vector of allocation of the fortune to each asset class $1, \ldots, p$, and $R_t^{(\cdot)} = \left(R_t^{(1)}, \ldots, R_t^{(p)}\right)'$ is the vector of returns of each asset at time t. We can decompose $\sum_R^{(t)}$ into $\alpha' \Gamma_R^{(t)} \alpha$, where $\Gamma_R^{(t)}$ is the variance matrix of $R_t^{(\cdot)}$. Thus

$$\begin{aligned} \operatorname{Var}\{L(R,C)\} &\approx \sum_{t=1}^{T} \alpha' \Gamma_R^{(t)} \alpha \left(\frac{\partial L}{\partial R_t}\right)^2 + \sum_{t=1}^{T} \Sigma_C^{(t)} \left(\frac{\partial L}{\partial C_t}\right)^2 \\ &= \sum_{t=1}^{T} \sum_{j=1}^{p} \alpha_j \Gamma_R^{(t)} \alpha \left(\frac{\partial L}{\partial R_t}\right)^2 + \sum_{t=1}^{T} \Sigma_C^{(t)} \left(\frac{\partial L}{\partial C_t}\right)^2 \end{aligned} \tag{4.15}$$

Therefore, the contribution to the variance

1. of the return of the asset at year is $\alpha_j \Gamma_R^{(t)} \alpha \left(\frac{\partial L}{\partial R_t}\right)^2$;
2. of the return of the asset from year to is $\sum_{t=t_0}^{t_1} \alpha_j \Gamma_R^{(t)} \alpha \left(\frac{\partial L}{\partial R_t}\right)^2$;
3. of the portfolio return at year is $\sum_{j=1}^{p} \alpha_j \Gamma_R^{(t)} \alpha \left(\frac{\partial L}{\partial R_t}\right)^2$; and
4. of the cash flow at year is $\sum_{t=1}^{T} \Sigma_C^{(t)} \left(\frac{\partial L}{\partial C_t}\right)^2$.

Every combination is possible. The relative contributions are obtained by dividing the contribution by the total sum. For example, the relative contribution to the return of the asset j at year t is

$$\alpha_j \Gamma_R^{(t)} \alpha \left(\frac{\partial L}{\partial R_t}\right)^2 \Big/ \operatorname{Var}\{L(L,C)\}$$

4.9 Example

Let's assume that the portfolio allocation is that of the index Pictet BVG/LPP 25:

Table 4.6

CHF Bonds	World Bonds	Equities Switzerland	Equities World	Real Estate Switzerland	Real Estate World	Hedge Funds	Private Equity
40.0%	25.0%	7.5%	12.5%	7.5%	2.5%	2.5%	2.5%

Let's assume furthermore that the annualized market hypotheses are given in the two following tables:

Table 4.7

	Annualized Expected Return	Annualized Expected Volatility
CHF Bonds	4.5%	5.4%
World Bonds	4.5%	5.5%
Equities Switzerland	8.0%	17.0%
Equities World	5.0%	18.0%
Real Estate Switzerland	6.0%	8.0%
Real Estate World	6.0%	25.0%
Hedge Funds	6.0%	7.0%
Private Equity	7.0%	25.0%

TABLE 4.8

Correlation Matrix	CHF Bonds	World Bonds	Equities Switzerland	Equities World	Real Estate Switzerland	Real Estate World	Hedge Funds	Private Equity
CHF Bonds	1.00							
World Bonds	0.60	1.00						
Equities Switzerland	0.10	0.05	1.00					
Equities World	0.06	0.07	0.80	1.00				
Real Estate Switzerland	0.34	0.23	0.34	0.28	1.00			
Real Estate World	0.16	0.16	0.64	0.80	0.33	1.00		
Hedge Funds	0.12	0.23	0.37	0.45	0.29	0.40	1.00	
Private Equity	0.03	-0.01	0.60	0.80	0.27	0.68	0.43	1.00

Furthermore, let's assume that the cash flows are structured according the following table.

TABLE 4.9

t	Expected Value	Volatility
1	100	5
2	80	4
3	60	3
4	40	2
5	20	1

Then the covariance matrix $\Gamma_R^{(t)}$ is constant in time and given below.

TABLE 4.10

Covariance Matrix	CHF Bonds	World Bonds	Equities Switzerland	Equities World	Real Estate Switzerland	Real Estate World	Hedge Funds	Private Equity
CHF Bonds	0.0029	0.0018	0.0009	0.0006	0.0015	0.0022	0.0005	0.0004
World Bonds	0.0018	0.0030	0.0005	0.0007	0.0010	0.0022	0.0009	-0.0001
Equities Switzerland	0.0009	0.0005	0.0289	0.0245	0.0046	0.0272	0.0044	0.0255
Equities World	0.0006	0.0007	0.0245	0.0324	0.0040	0.0360	0.0057	0.0360
Real Estate Switzerland	0.0015	0.0010	0.0046	0.0040	0.0064	0.0066	0.0016	0.0054
Real Estate World	0.0022	0.0022	0.0272	0.0360	0.0066	0.0625	0.0070	0.0425
Hedge Funds	0.0005	0.0009	0.0044	0.0057	0.0016	0.0070	0.0049	0.0075
Private Equity	0.0004	-0.0001	0.0255	0.0360	0.0054	0.0425	0.0075	0.0625

For example, the covariance between CHF Bonds $(j=1)$ and World Bonds $(j=2)$ is computed as $\sigma_1\sigma_2\rho_{1,2} = 0.054\times 0.055\times 0.6 = 0.018$. Cross products between covariances and portfolio allocations are given in the table below.

TABLE 4.11

j	$\left\{\Gamma_R^{(t)}\alpha\right\}_j$	$\alpha_j\left\{\Gamma_R^{(t)}\alpha\right\}_j$
CHF Bonds	0.00194	0.00078
World Bonds	0.00174	0.00044
Equities Switzerland	0.00749	0.00056
Equities World	0.00854	0.00107
Real Estate Switzerland	0.00251	0.00019
Real Estate World	0.01125	0.00028
Hedge Funds	0.00205	0.00005
Private Equity	0.00976	0.00024

The derivatives are then given in the following table.

TABLE 4.12

t	$E(C_t)$	$E(R_t)$	$\frac{C_t}{\prod_{j=1}^{t}(1+R_j)}$	$\frac{\partial L}{\partial R_t}$	$\frac{\partial L}{\partial C_t}$
1	100	5.08%	95.17	-254.84	0.95
2	80	5.08%	72.46	-164.27	0.91
3	60	5.08%	51.72	-95.31	0.86
4	40	5.08%	32.81	-46.09	0.82
5	20	5.08%	15.61	-14.86	0.78

Therefore, the contributions to the variance are

TABLE 4.13

j/t	1	2	3	4	5	
CHF Bonds	50.38	20.93	7.05	1.65	0.17	80.18
World Bonds	28.26	11.74	3.95	0.92	0.10	44.97
Equities Switzerland	36.46	15.15	5.10	1.19	0.12	58.03
Equities World	69.30	28.80	9.69	2.27	0.24	110.29
Real Estate Switzerland	12.24	5.08	1.71	0.40	0.04	19.47
Real Estate World	18.26	7.59	2.55	0.60	0.06	29.07
Hedge Funds	3.33	1.38	0.47	0.11	0.01	5.30
Private Equity	15.84	6.58	2.22	0.52	0.05	25.22
Cash Flows	22.64	13.13	6.69	2.69	0.61	45.76
	256.72	110.38	39.43	10.35	1.41	418.28

The total variance amounts to $\text{Var}(L) = 50.38 + \ldots + 0.61 = 418.28$ whereas the standard deviation amounts to $\sigma(L) = \sqrt{418.28} = 20.45$. Finally, the relative contributions are given in the following table.

TABLE 4.14

j/t	1	2	3	4	5	
CHF Bonds	12.04%	5.00%	1.68%	0.39%	0.04%	19.17%
World Bonds	6.76%	2.81%	0.94%	0.22%	0.02%	10.75%
Equities Switzerland	8.72%	3.62%	1.22%	0.29%	0.03%	13.87%
Equities World	16.57%	6.88%	2.32%	0.54%	0.06%	26.37%
Real Estate Switzerland	2.93%	1.22%	0.41%	0.10%	0.01%	4.66%
Real Estate World	4.37%	1.81%	0.61%	0.14%	0.01%	6.95%
Hedge Funds	0.80%	0.33%	0.11%	0.03%	0.00%	1.27%
Private Equity	3.79%	1.57%	0.53%	0.12%	0.01%	6.03%
Cash Flows	5.41%	3.14%	1.60%	0.64%	0.15%	10.94%
	61.37%	26.39%	9.43%	2.47%	0.34%	100.00%

Therefore, the contributions to the variance are obtained by summing the terms in the previous table.

For examples, the contribution

1. of the return of the Swiss bonds at year 1 is 12.04%;
2. of the return of the Swiss bonds from year 1 to 3 is $12.04\% + 5.00\% + 1.68\% = 18.73\%$;
3. of the portfolio return at year 1 is $12.04\% + \ldots + 3.79\% = 55.96\%$;
4. of the cash flow at year 4 is 0.64%;

5. of the year 3 is $1.68\%+\ldots+1.60\%=9.43\%$.

4.10 The Distribution of $L(R,C)$

In the following, we assume that $L(R,C)$ is distributed according to a log-normal distribution, then the VaR and expected shortfall can be computed accordingly. Recall that if L is log-normally distributed then $\ln L$ is normally distributed with expected value μ and variance σ^2. Thus, the expected value and the variance of L are respectively $E(L)=e^{\mu+\sigma^2/2}$ and $\mathrm{Var}(L)=(e^{\sigma^2}-1)e^{2\mu+\sigma^2}$.

Furthermore, the VaR at level α and the expected shortfall equal respectively

$$\mathrm{VaR}_\alpha(L)=e^{\mu+\sigma\Phi^{-1}(\alpha)} \tag{4.16}$$

and

$$\mathrm{ES}_\alpha(L)=\left(1-\alpha\right)^{-1}e^{\mu+\sigma^2/2}\Phi\left(\alpha-\Phi^{-1}\left(\alpha\right)\right). \tag{4.17}$$

The risk here is associated with high values of L, in other words $P\left\{L>\mathrm{VaR}_\alpha\left(L\right)\right\}=\alpha$ and $\mathrm{ES}_\alpha(L)=E\left[L\middle|L\geq VaR_\alpha\left(L\right)\right]$. In order to apply the previous approximations, it suffices to compute the derivatives of $\ln L$. To do so, one can note that for each risk factor x

$$\frac{\partial\ln L}{\partial x}=\frac{1}{L}\frac{\partial L}{\partial x}$$

and

$$\frac{\partial^2\ln L}{\partial x^2}=\frac{1}{L}\left\{\frac{\partial^2 L}{\partial x^2}-\frac{1}{L}\left(\frac{\partial L}{\partial x}\right)^2\right\}.$$

4.11 Example

Using the values computed in a previous example, we have

Table 4.15

t	$\frac{\partial\ln L}{\partial R_t}$	$\frac{\partial^2\ln L}{\partial R_t^2}$	$\frac{\partial\ln L}{\partial C_t}$	$\frac{\partial^2\ln L}{\partial C_t^2}$
1	–0.97	0.94	0.003561	–0.000013
2	–0.61	0.79	0.003392	–0.000012
3	–0.35	0.54	0.003215	–0.000010
4	–0.17	0.29	0.003047	–0.000009
5	–0.05	0.10	0.002888	–0.000008

Table 4.16

t	$\sigma^2(R_t)\frac{\partial^2\ln L}{\partial R_t^2}$	$\sigma^2(C_t)\frac{\partial^2\ln L}{\partial C_t^2}$	$\sigma^2(R_t)\left(\frac{\partial\ln L}{\partial R_t}\right)^2$	$\sigma^2(C_t)\left(\frac{\partial\ln L}{\partial C_t}\right)^2$
1	0.00151	–0.00032	0.00151	0.00032
2	0.00285	–0.00018	0.00135	0.00018
3	0.00267	–0.00009	0.00061	0.00009
4	0.00144	–0.00004	0.00014	0.00004
5	0.00049	–0.00001	0.00001	0.00001

Furthermore

$$\ln L=\left(\mu_R,\mu_C\right)=\ln 272.62=5.61$$

and

$$\mathrm{Tr}\left\{\Sigma\nabla^2 \ln L\right\} = \sum_{t=1}^{T}\Sigma_R^{(t)}\frac{\partial^2 \ln L}{\partial R_t^2}+\sum_{t=1}^{T}\Sigma_C^{(t)}\frac{\partial^2 \ln L}{\partial C_t^2} = 0.0083.$$

Finally $E(\ln L) = 5.61 + 0.0083/2 = 5.62$ and $\sigma^2(\ln L) \approx 0.0042$ so that $\sigma(\ln L) \approx 0.065$. Using the log-normal formula,

$$E(L) = \exp(5.61 + 0.0043/2) = 274.35$$

and

$$\sigma(L) = \sqrt{\left(\exp(0.0042)-1\right)\exp(5.62+0.0042)} = 17.94.$$

The following table computes VaR and ES at levels 90%, 95% and 99.5%.

TABLE 4.17

α	$\Phi^{-1}(\alpha)$	VaR	ES
90.0%	1.28	297.67	307.14
95.0%	1.64	304.82	313.35
99.5%	2.58	323.94	330.76

4.12 LOGARITHM OR NOT LOGARITHM?

In the previous paragraphs, the variance of L was computed under two different methodologies: the first one directly, and the second one from the logarithm of L and the hypothesis of log-normality. The finality of these two methodologies is different:

1. the Taylor expansion of L leads to the decomposition of its variance into risk factor contributions;

2. the Taylor expansion of $\ln L$ leads to its expectation and its variance, from which the VaR and the ES for L can be computed, under the hypothesis of log-normality.

The variances obtained under 1 and 2 are different, though hopefully close. The computation of the variance under point 1 is more coherent with the hypothesis of log-normality, and should be closer to reality. The computation under point 2 is a necessity because an additive decomposition of the variance of into risk factor contributions would lead to a multiplicative decomposition of the variance of L, which would be very difficult to interpret.

4.13 THE DISTRIBUTION OF THE FUNDING RATIO

The funding ratio is defined as $\rho = F/L$, and, thus, the logarithm of the funding ratio is $\ln\rho = \ln F - \ln L$. It follows that, if L is distributed according to a log-normal distribution, so is also ρ. Furthermore, the expected value of $\ln\rho$ is $E(\ln\rho) = \ln F - E(\ln L)$ and its variance $\sigma(\ln\rho) = \sigma(\ln L)$.

For the VaR and the ES, one must remark that an upside risk for the liabilities corresponds to a downside risk for the funding ratio. Therefore, the risk here is associated with low values of ρ, in other words $P\left\{\rho \le \mathrm{VaR}_\alpha\left(\rho\right)\right\} = 1-\alpha$ and $\mathrm{ES}_\alpha\left(\rho\right) = E\left[\rho \middle| \rho \le \mathrm{VaR}_\alpha\left(\rho\right)\right]$. One can compute

$$\mathrm{VaR}_\alpha(\rho) = e^{\mu+\sigma\Phi^{-1}(1-\alpha)} \tag{4.18}$$

and

$$\mathrm{ES}_\alpha(\rho) = \left(1-\alpha\right)^{-1} e^{\mu+\sigma^2/2}\Phi\left(\Phi^{-1}\left(1-\alpha\right)-\alpha\right). \tag{4.19}$$

4.14 EXAMPLE

In the previous example we have, approximately, $E(\ln L) = 5.61$ and $\sigma(\ln L) = 0.065$. Therefore, with a fortune fixed at F, we have $E(\ln\rho) = \ln 280 - E(\ln L) = 0.023$ and $\sigma(\ln\rho) = 0.065$.

Using the log-normal formula, we find $E(\rho) = 102.5\%$ and $\sigma(\rho) = 6.7\%$. The following table computes VaR and ES figures at levels 90%, 95% and 99.5%.

TABLE 4.18

α	$\Phi^{-1}(1-\alpha)$	VaR	ES
90.0%	-1.28	93.91%	90.95%
95.0%	-1.64	91.47%	89.09%
99.5%	-2.58	85.23%	84.26%

5 FLUCTUATION RESERVES

Fluctuation reserves are that part of liabilities designed to absorb future fluctuations of the asset values due market uncertainty. They are defined as reserves above technical liabilities. The fluctuation reserves objective, FRO, is related to a risk level and corresponds to the excess above technical liabilities of the risk measures developed in the risk analysis

$$\mathrm{FRO}_\alpha = \mathrm{Risk}_\alpha - L \tag{5.1}$$

where Risk_α is a risk measure (VaR or ES) at level α and L is the technical liabilities. The fluctuation reserves objective can also be expressed relatively to the technical liabilities, rFRO

$$\mathrm{rFRO}_\alpha = \frac{\mathrm{Risk}_\alpha - L}{L}. \tag{5.2}$$

5.1 EXAMPLE

Following the previous examples, with technical liabilities at 280.20, for a technical rate at 3%, then the FRO and the rFRO are given in the following table.

TABLE 5.1

α	VaR	FRO to VaR	rFRO to VaR	ES	FRO to ES	rFRO to ES
90.0%	297.34	17.15	6.12%	307.14	26.94	9.61%
95.0%	303.86	23.67	8.45%	313.35	33.16	11.83%
99.5%	320.57	40.37	14.41%	330.76	50.57	18.05%

6 CONCLUSIONS

In this document, the liabilities are viewed as the amount needed today to meet the commitments of the pension fund toward its members. This amount thus depends on the strategy the pension fund applies, and therefore on the unknown future portfolio returns as well as the unknown future cash flows.

Incorporating, the resulting uncertainty is a matter of mathematical modeling. In this document the lognormal model is used to:

1. compute value at risk and expected shortfall of the pension fund liabilities and of the funding ratio;
2. decompose the risk on the liabilities into various risk factors (market sectors, cash flows and time); and
3. compute fluctuation reserve objective relative to the technical liabilities.

The ultimate aim of this document is to explain the formulas used in the Solvenswiss. It does not present the justification to the log-normal hypothesis, neither does it discuss the issue of market models (Gaussian or GARCH) or the issue relating to the choice of particular market sectors (mapping). These will be the issues for further publications.